

HÖLDER REGULARITY OF GENERIC MANIFOLD

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Dedicated to Professor Józef Siciak for his 85th birthday

Abstract : In this paper we study Hölder continuity of the pluricomplex Green function with logarithmic growth at infinity of a smooth generic submanifold of \mathbb{C}^n . In particular we prove that the pluricomplex Green function of any C^2 -smooth generic compact submanifold of \mathbb{C}^n (without boundary) is Lipschitz continuous in \mathbb{C}^n .

Key words : Generic manifold, attached analytic discs, plurisubharmonic Green function, pluripolar sets, pluriregular sets, Hölder continuity.

AMS Classification : 32U05, 32U15, 32U35, 32E30, 32V40

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULT

Real m -planes $\Pi \subset \mathbb{C}^n$, $\dim_R \Pi = m$, $m \in \mathbb{N}^+$, which are not contained in any proper complex subspace of \mathbb{C}^n are important in complex analysis and pluripotential theory. The \mathbb{C} -hull of such plane Π is equal to all \mathbb{C}^n i.e. $\Pi + J\Pi = \mathbb{C}^n$ (J is the standard complex structure on \mathbb{C}^n) and any non empty open subset of Π is non pluripolar in \mathbb{C}^n . Such planes are called *generic* (real) subspaces of \mathbb{C}^n . Correspondingly, a real smooth submanifold $M \subset \mathbb{C}^n$ is said to be *generic* if for each $z \in M$, its real tangent space $T_z M$ is a generic subspace of \mathbb{C}^n i.e. $T_z M + JT_z M = \mathbb{C}^n$. Such submanifold has real dimension $m \geq n$. The case of minimal dimension $\dim M = n$ is the most relevant for our concern. In this case for each $z \in M$, the tangent space $T_z M$ does not contain any complex line i.e. $T_z M \cap JT_z M = \{0\}$ and M is said to be *totally real*.

Observe that any smooth Jordan curve in \mathbb{C} is totally real, hence any product of n smooth Jordan curves in \mathbb{C} is a smooth compact totally real submanifold of dimension n in \mathbb{C}^n . Moreover the class of smooth compact totally real submanifolds of dimension n in \mathbb{C}^n is stable under small C^2 -perturbations.

Generic submanifolds of \mathbb{C}^n play an important role in Complex Analysis and Pluripotential Theory (see [Pi74], [KC76], [Sa76], [C92], [EW10], [SZ12]).

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In our previous paper [SZ12], we used the method of attached analytic discs to investigate non plurithinness of generic submanifolds of \mathbb{C}^n . We proved in [SZ12], that subsets of full measure in a generic C^2 -smooth submanifold are non-plurithin at any point.

Here we continue our investigations concerning the *pluripotential* properties (*pluripolarity*, *pluriregularity*) of generic submanifolds in \mathbb{C}^n by studying Hölder continuity of their pluricomplex Green functions.

All these properties can be expressed in terms of the pluricomplex Green function defined as follows.

Given a (bounded) subset $E \Subset \mathbb{C}^n$, we define its pluricomplex Green function as follows:

$$V_E(z) := \sup\{u(z) : u \in \mathcal{L}(\mathbb{C}^n), u|_E \leq 0\},$$

where $\mathcal{L}(\mathbb{C}^n)$ is the Lelong class of *psh* functions u in \mathbb{C}^n with logarithmic growth at infinity i.e. $\sup\{u(z) - \log^+(z) : z \in \mathbb{C}^n\} < +\infty$ (see [Sic62], [Sic81], [Za76], [Kl]).

Our main result is the following.

Main theorem. *Let $M \subset \mathbb{C}^n$ be a C^2 -smooth generic compact submanifold without boundary. Then its pluricomplex Green function V_M is Lipschitz continuous in \mathbb{C}^n .*

This theorem is concerned with compact submanifolds without boundary. In Section 3, we will consider the more general case of a C^2 -smooth generic submanifold and prove that its extremal function is Lipschitz near each of its compact subsets (see Theorem 5.1). In the last section we consider the case of a compact C^2 -smooth generic submanifold with boundary and discuss the Hölder continuity property of its pluricomplex Green function.

From Lipschitz continuity or more generally the Hölder continuity of the pluricomplex Green function $V_E^*(z)$ of a compact set $E \subset \mathbb{C}^n$, it follows that the compact set E satisfies the following Markov's inequality: there exists positive constants $A, r > 0$ such that

$$\|\nabla P(z)\|_E \leq Ad^r \|P(z)\|_E, \quad z \in \mathbb{C}^n$$

for any polynomial P of degree d .

This inequality plays an important role in approximation theory, gives sharp inequalities for polynomials and is useful for constructing continuous extension operators for smooth functions from subsets of \mathbb{R}^n to \mathbb{C}^n (see [PP86], [Ze93]). On the other hand, Complex Dynamic gives a lot of examples of compact subsets for which the pluricomplex Green function $V_E^*(z)$ is Hölder continuous (see [FS92, K95, Ks97]).

More recently an important result of C.T. Dinh, V. A. Nguyen and N. Sibony shows that the Monge-Ampère measure of a Hölder continuous plurisubharmonic function is a *moderate measure* (see [DNS]). In particular the equilibrium Monge-Ampère measure $\mu_E := (dd^c V_E)^n$ of a compact subset whose pluricomplex Green function $V_E^*(z)$ is Hölder continuous is

a moderate measure, which means that it satisfies the following uniform version of Skoda's integrability theorem: for any compact family \mathcal{U} of *psh* functions in a neighborhood of a given ball $\mathbb{B} \subset \mathbb{C}^n$, there exists $\varepsilon > 0$ and a constant $C > 0$ such that

$$\int_{\mathbb{B}} e^{-\varepsilon u} d\mu_E \leq C, \forall u \in \mathcal{U}.$$

From this property, it follows that the equilibrium measure μ_E is "well dominated" by the Monge-Ampère capacity (see [Ze01]), in the sense that for any given ball $\mathbb{B} \subset \mathbb{C}^n$, there is a constant $A > 0$ such that for any Borel set $S \subset \mathbb{B}$,

$$\mu_E(S) \leq A \exp \left(-A \text{cap}_{\mathbb{B}}(S)^{-1/n} \right),$$

where $\text{cap}_{\mathbb{B}}(S)$ is the Monge-Ampère capacity ([BT82]).

This property turns out to play an important role in the theory of complex Monge-Ampère equations as was discovered by S. Kolodziej (see [Ko98], [Ce98], [GZ07]).

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2. DEFINITIONS AND PRELIMINARIES

Let us recall the following definitions

Definition 2.1. 1. We say that a subset $P \subset \mathbb{C}^n$ is pluripolar if there is a plurisubharmonic (*psh*) function $u : u \not\equiv -\infty$ but $u|_P \equiv -\infty$.
 2. We say that E is pluriregular if its pluricomplex Green function satisfies $V_E^*(z)|_E \equiv 0$ i.e. $V_E = V_E^*$ on \mathbb{C}^n .

Observe that any pluriregular set is non-pluripolar. It is well-known that if E is non-pluripolar then $V_E^* \in \mathcal{L}(\mathbb{C}^n)$. Moreover if E is pluriregular compact set then $V_E = V_E^*$ is continuous in \mathbb{C}^n (see [Sic81]).

On the other hand, we know from ([BT82]) that if E is non-pluripolar then the locally bounded *psh* function V_E^* satisfies the following complex Monge-Ampère equation

$$(dd^c V_E)^n = 0, \quad \text{on } \mathbb{C}^n \setminus \overline{E},$$

which means that the equilibrium measure of E defined as

$$\mu_E := (dd^c V_E^*)^n$$

is a Borel measure supported in the closed set \overline{E} .

Here we will introduce the following important notion.

Definition 2.2. *We say that a set E is Λ_α -pluriregular, $\alpha > 0$, if for every compact $K \subset E$ there exist a constant $A = A_K > 0$ and a neighborhood $O = O_K$ of K such that*

$$(2.1) \quad V_E^*(z) \leq Ad^\alpha(z, K), \quad \forall z \in O,$$

where d is the Euclidean distance in \mathbb{C}^n .

Roughly speaking, this definition means that the pluricomplex Green function V_E of the set E is Hölder continuous near any compact subset $K \subset E$. The following observation, which is essentially due to Z. Błocki, shows that if the set itself E is compact, the definition means that its pluricomplex Green function is Hölder continuous (see [Sic97]).

Lemma 2.3. *If $E \subset \mathbb{C}^n$ is a Λ_α -pluriregular compact set then its pluricomplex Green function V_E is Hölder continuous of order α globally in \mathbb{C}^n i.e. for any $z, w \in \mathbb{C}^n$, we have*

$$|V_E(z) - V_E(w)| \leq A|z - w|^\alpha.$$

Proof. Observe that V_E^* has a logarithmic growth at infinity. Therefore if E is a Λ_α -pluriregular compact set then its pluricomplex Green function V_E satisfies (2.1) for all $z \in \mathbb{C}^n$ i.e. for some constant $A > 0$ we have

$$(2.2) \quad V_E(z) \leq Ad^\alpha(z, E), \quad \forall z \in \mathbb{C}^n.$$

To prove that V_E is Hölder continuous of order α globally in \mathbb{C}^n , fix $h \in \mathbb{C}^n$ such that $|h| < \delta$ and observe, that for any $z \in E$, $d(z + h, E) \leq \delta^\alpha$, which implies by the Hölder condition (2.2) that for any $z \in E$, $V_E(z + h) \leq A\delta^\alpha$. Therefore the function defined by $u(z) := V_E(z + h) - A\delta^\alpha$ is a plurisubharmonic function such that $u \in \mathcal{L}(\mathbb{C}^n)$ and $u \leq 0$ on E . By the definition of V_E , we conclude that $u \leq V_E(z)$ for any $z \in \mathbb{C}^n$, which implies that V_E is Hölder continuous. \square

3. ANALYTIC DISCS ATTACHED TO GENERIC MANIFOLDS

3.1. Construction of attached analytic discs. Let $\mathbb{U} := \{\zeta \in \mathbb{C} : |\zeta| < 1\}$ be the open unit disc and $\mathbb{T} := \partial\mathbb{U}$ the unit circle. An analytic disc of \mathbb{C}^n is a continuous function $f : \overline{\mathbb{U}} \rightarrow \mathbb{C}^n$, which is holomorphic on \mathbb{U} . Let $M \subset \mathbb{C}^n$ be a given subset of \mathbb{C}^n and $\gamma \subset \overline{\mathbb{U}}$ a given connected subset of the closed disc $\overline{\mathbb{U}}$. We say that the analytic disc f is attached to M along γ if $f(\gamma) \subset M$.

If $f : \overline{\mathbb{U}} \rightarrow \mathbb{C}^n$ is an analytic disc and F is a holomorphic function on a neighborhood D of $f(\overline{\mathbb{U}})$, then $F \circ f$ is a holomorphic function on the unit

disc \mathbb{U} . If u is a plurisubharmonic function on D , then $u \circ f$ is a subharmonic function on \mathbb{U} . Therefore analytic discs enable us to reduce multidimensional complex problems to corresponding one dimensional complex problems.

In the proof of our theorem, we need a smooth family of analytic disks. We will use Bishop's equation for construct such a family (see [B65], [Pi74]). Let M be a totally real submanifold of dimension n given locally by the following equation

$$M := \{z = x + y \in B \times \mathbb{R}^n : y = h(x)\},$$

where $B \subset \mathbb{R}^n$ is a ball of center 0 and $h : B \rightarrow \mathbb{R}^n$ a smooth map, such that

$$h(0) = 0 \quad \text{and} \quad Dh(0) = 0.$$

Let $v(\tau) : \mathbb{T} \rightarrow \mathbb{R}^+$ a C^∞ function on the unit circle \mathbb{T} such that

$$v|_{\{e^{i\theta} : \theta \in (0, \pi)\}} = 0 \quad \text{and} \quad v|_{\{e^{i\theta} : \theta \in (\pi, 2\pi)\}} > 0.$$

Assume that there exists a continuous mapping $X : \mathbb{T} \rightarrow \mathbb{R}^n$ which is a solution of the following Bishop equation

$$(3.1) \quad X(\tau) = c - \Im(h \circ X + tv)(\tau), \quad \tau \in \mathbb{T},$$

where $(c, t) \in Q = Q_c \times Q_t \subset \mathbb{R}^n \times \mathbb{R}^n$ is a fixed parameter and \Im is the harmonic conjugate operator defined by the Schwarz integral formula

$$(3.2) \quad \Im(X)(\zeta) = \frac{1}{2\pi} \int_T X(\tau) \operatorname{Im} \frac{e^{i\tau} + \zeta}{e^{i\tau} - \zeta} d\tau, \quad \zeta = re^{i\theta},$$

normalized by the condition

$$\Im X(0) = 0.$$

We will consider the unique harmonic extension $X(\zeta)$ of the mapping X to the unit disk \mathbb{U} . Then the following mapping

$$(3.3) \quad \begin{aligned} \Phi(c, t, \zeta) &:= X(c, t, \zeta) + i[h^*(c, t, \zeta) + tv(\zeta)] = \\ &= c + i\{h^*(c, t, \zeta) + tv(\zeta) + i\Im[h^*(c, t, \zeta) + tv(\zeta)]\} \end{aligned}$$

provides a family of analytic disks $\Phi(c, t, \zeta) : \overline{\mathbb{U}} \rightarrow \mathbb{C}^n$ such that

$$(3.4) \quad \forall (c, t) \in Q, \forall \tau \in \gamma, \Phi(c, t, \tau) \in M.$$

Here $X(c, t, \zeta)$, $h^*(c, t, \zeta)$ and $v(\zeta)$ are harmonic extensions of $X(c, t, \tau)$, $h \circ X(c, t, \tau)$ and $v(\tau)$ to the unit disk \mathbb{U} respectively.

We need a smooth family of disks $\Phi(c, t, \zeta)$. Many constructions of analytic discs attached to generic manifolds along a part the circle have been given by many different authors, depending on the smoothness properties of the manifold (see [Pi74], [75], [Sa76], [C92]). The most general and sharp result was proved by B.Coupet [C92]:

Theorem ([C92]). *Let $p > 2n + 1, q \geq 1$ be integers and $h \in C^q(B)$. Then there exist a constant $\delta_0 > 0$ depending on h and p such, that for*

arbitrary C^q -smooth mapping $k(c, t, \tau) : \mathbb{R}^{2n+1} \rightarrow \mathbb{R}^n$, with compact support and $\|k\|_{W^{q,p}} \leq \delta_0$, the equation

$$(3.5) \quad u = -\mathfrak{S}(h \circ u) + k$$

has a unique solution $u \in W^{q,p}(\mathbb{T} \times \mathbb{R}^{2n})$.

Moreover, the harmonic extensions of u and $h \circ u$ to the unit disk \mathbb{U} belong to $C^q(\mathbb{U} \times \mathbb{R}^{2n})$.

Let now $h \in C^q(\mathbb{B})$. Observe that Bishop's equation (3.1) is a particular case of the equation (3.5). Therefore, from the theorem of Coupet and Sobolev's embedding theorem $W^{q,p} \subset C^{q-1}$, it follows that for a small enough neighborhood $Q \ni 0$, $(c, t) \in Q$, the Bishop equation (3.1) has unique solution $X(\tau, c, t) : X, h \circ X \in C^{q-1}(\overline{U} \times Q) \cap C^q(U \times Q)$. Note that the operator $\mathfrak{S} : W^{q,p} \rightarrow W^{q,p}$ is continuous.

Therefore, for a C^2 -smooth generic submanifold $M \subset \mathbb{C}^n$, we obtain a smooth family of disks (3.3), attached to M , such that

$$\|\mathfrak{S}X\|_1 \leq A \|X\|_1, \|\mathfrak{S}h \circ X\|_1 \leq A \|h \circ X\|_1,$$

where A is a constant and $\|\cdot\|_1$ is the C^1 -norm in $\tau \in T$.

3.2. Harmonic measure of boundary set of the unit disk. For arbitrary $\gamma \subset \mathbb{T}$ we put $\aleph(\gamma, \mathbb{U})$ - class of functions

$$\{u(\zeta) : u \in \text{sh}(\mathbb{U}) \cap C(\overline{\mathbb{U}}), u|_{\mathbb{U}} < 0, u|_{\gamma} \leq -1\},$$

and set

$$\omega(\zeta, \gamma, \mathbb{U}) = \sup\{u(\zeta) : u \in \aleph\}, \quad \zeta \in \mathbb{U}.$$

Then (negative of) the upper semi-continuous regularisation $\omega^*(\zeta, \gamma, \mathbb{U})$ is called the *harmonic measure* of γ with respect to \mathbb{U} at the point ζ . The function ω^* is the unique solution of the Dirichlet problem:

$$\Delta \omega^* = 0, \quad \omega^*|_{\mathbb{T}} = -\chi_{\gamma},$$

where χ_{γ} is the characteristic function of γ . By Poisson formula

$$\omega^*(\zeta, \gamma, \mathbb{U}) = -\frac{1}{2\pi} \int_{\mathbb{T}} \chi_{\gamma}(\tau) \operatorname{Re} \frac{e^{i\tau} + \zeta}{e^{i\tau} - \zeta} d\tau, \quad \zeta = re^{i\theta}.$$

For $\gamma = \{e^{i\varphi} : 0 \leq \varphi \leq \pi\}$ the harmonic measure ω^* can be expressed as follows.

$$(3.6) \quad \omega^*(\zeta, \gamma, \mathbb{U}) = \frac{1}{\pi} \arg i \frac{1 - \zeta}{1 + \zeta} - 1.$$

Let us define the sector at the point $1 = e^{i \cdot 0} \in \overline{\mathbb{U}}$ as follows

$$\Omega_{0,\alpha} = \cup \{l \cap \overline{\mathbb{U}} : l \ni 1, \pi/2 \leq \arg l \leq \pi/2 + \alpha\},$$

where l stands for a real line passing through the point 1 and $0 \leq \alpha \leq \pi/2$ is fixed. The sector $\Omega_{a,\alpha}$ at the point $e^{ia} \in \overline{\mathbb{U}}$ can be defined in the same way. From (3.6) it follows clearly that

$$(3.7) \quad \omega^*(\zeta, \gamma, \mathbb{U}) \leq -1 + \alpha/\pi, \quad \forall \zeta \in \Omega_{0,\alpha}.$$

A sector $\Omega_{a,\alpha}$ at the point e^{ia} is said to be *admissible* if $\Omega_{a,\alpha} \cap \partial\mathbb{U} \subset \gamma$. From the last fact, we deduce the following statement.

Lemma 3.1. *Let $\gamma = \text{arc}[e^{ia}, e^{ib}] \subset \mathbb{T}$, $0 \leq a < b \leq 2\pi$, be an arbitrary arc on \mathbb{T} , and let $\Omega_{a,\alpha}$, be an admissible sector at the point e^{ia} . Then $\omega^*(\zeta, \gamma, \mathbb{U})$ is Λ_1 -continuous in $\mathbb{U} \cup \mathbb{T} \setminus \{e^{ia}, e^{ib}\}$, $\omega^*|_{\gamma^\circ} \equiv -1$, $\omega^*|_{\mathbb{T} \setminus \gamma} \equiv 0$ and ω^* satisfies (3.7) in $\Omega_{a,\alpha}$.*

Here γ° denote the interior of the arc γ . We note that if $\gamma_0 \Subset \gamma$ in an arc with non empty interior, then there exist $\alpha = \alpha(\gamma_0, \gamma) > 0$ such that $\Omega_{\tau,\alpha}$ is admissible for every $\tau \in \gamma_0$.

4. TRANSVERSALITY OF ATTACHED DISCS TO A GENERIC MANIFOLD.

It is clear that the family of analytic discs constructed above

$$\Phi(c, t, \zeta) = X(c, t, \zeta) + i(h^*(c, t, \zeta) + tv(\zeta)),$$

for $(c, t) \in Q = Q_c \times Q_t, \zeta \in \bar{\mathbb{U}}$, satisfies the following properties:

$$(4.1) \quad X(c, t, \tau) = c - \Im(h \circ X(c, t, \tau) + tv(\tau)), \quad (c, t) \in Q, \tau \in \partial\mathbb{U}.$$

$$(4.2) \quad h^*(c, t, \tau) = h \circ X(c, t, \tau), \quad (c, t) \in Q, \tau \in \partial\mathbb{U}$$

$$(4.3) \quad \begin{aligned} X(c, 0, \zeta) &\equiv c, h^*(c, 0, \zeta) \equiv h(c) \text{ so that} \\ \Phi(c, 0, \zeta) &\equiv c + ih(c) \in M, c \in Q_c \end{aligned}$$

$$(4.4) \quad X(c, t, 0) = \frac{1}{2\pi} \int_T X(c, t, \tau) d\tau \equiv c, \quad (c, t) \in Q.$$

$$(4.5) \quad \|X\| \leq O(\|c\| + \|t\|), \|D_\tau X\| \leq O(\|t\|).$$

Here and below $\|\cdot\|$ is Euclidean norm.

The following geometric transversality property will be crucial for the proof of our main theorem.

Lemma 4.1. *Let $\gamma_0 \subset \subset \gamma$ an arc with non empty interior. Then for small enough Q the attached disks $\Phi(c, t, \zeta)$, $t \neq 0$, for $\zeta \rightarrow \tau \in \gamma_0$ meet M transversally.*

Proof. For the normal derivative $D_{\vec{n}}$ at the points $\tau \in \gamma_0$ we have

$$\text{Im} D_{\vec{n}} \Phi(c, t, \tau) = D_{\vec{n}} h^*(c, t, \tau) + t D_{\vec{n}} v(\tau)$$

and

$$\left| \text{Im} D_{\vec{n}} \Phi(c, t, \tau) \right| \geq \|t\| b - O(\varepsilon) \|t\| = \|t\| (b - O(\varepsilon)),$$

where

$$b := \inf_{\gamma_0} \left| D_{\vec{n}} v(\tau) \right| > 0 \text{ and } \varepsilon = \sup \{ \|c\| + \|t\| : c \in Q_c, t \in Q_t \}.$$

It follows, that for $O(\varepsilon) < \frac{b}{2}$

$$(4.6) \quad \left| \operatorname{Im} D_{\vec{n}} \Phi(c, t, \tau) \right| \geq \|t\| b/2 \quad \forall \tau \in \gamma_0,$$

i.e. the disks $\Phi(c, t, \zeta)$ meet M for $\zeta \rightarrow \tau \in \gamma_0$ transversally. \square

Corollary 4.2. *Let $Q' = \{\|t\| = \sigma\} \subset Q_t$, where $\sigma > 0$. Then there exist a neighborhood $\Omega' \supset \gamma_0$ and a constant $C > 0$ such that*

$$(4.7) \quad \begin{aligned} d_{\mathbb{C}}(\zeta, \gamma_0) &\leq C d_{\mathbb{C}^n}[\Phi(c, t, \zeta), M] \\ d_{\mathbb{C}^n}[\Phi(c, t, \zeta), \Phi(c, t, \gamma_0)] &\leq C d_{\mathbb{C}^n}[\Phi(c, t, \zeta), M], \end{aligned}$$

$\forall \zeta \in \Omega = \overline{\Omega \cap \Omega'}$, $t \in Q'$, $c \in \overline{Q_c}$. Here $d_{\mathbb{C}}$ and $d_{\mathbb{C}^n}$ are Euclidean distances on \mathbb{C} and \mathbb{C}^n , respectively.

Proof. The statement clearly follows from (4.6), because for every fixed $t^0 \in Q'$, $c^0 \in \overline{Q_c}$ we can write (4.7), which then will be true in some neighborhoods $B_c \ni c^0$, $B_t \ni t^0$. \square

Lemma 4.3. *For every $\Omega' \supset \gamma_0$ and for every $Q'_t = \{\|t\| = \sigma\} \subset Q_t$, $\sigma > 0$ small enough the closed set $W = \{\Phi(c, t, \zeta) \in \mathbb{C}^n : c \in \overline{Q_c}, t \in Q'_t, \zeta \in \Omega = \overline{\Omega \cap \Omega'}\}$ contains the point $0 \in M$ in its interior in \mathbb{C}^n i.e. $0 \in \dot{W}$.*

Proof. By (4.3) $X(c, t, \zeta) \equiv c$ if $t = 0$. Since X is smooth, then for small enough fixed t^0 and for arbitrary fixed $\tau^0 \in \gamma_0$ the image $X(c, t^0, \tau^0) : c \in Q_c$ contains $0 \in \mathbb{R}^n$. It follows, that $0 \in W$. Moreover, $\dot{W} \neq \emptyset$ and if for some $\|t^0\| \leq \sigma$, $\zeta^0 \in \overline{U}$

$$(4.8) \quad x(c, t^0, \zeta^0) \in \frac{1}{2}Q_c, \text{ then } c \in Q_c$$

Now we assume by contradiction that $0 \in \partial W$. Then $\mathbb{C}^n \setminus W$ is open and contains 0 on its boundary. It is clear, that near 0 there exists a point $p^0 = (x^0, y^0) \in \partial \dot{W} \setminus M$ such, that $x^0 \in \frac{1}{2}Q_c$ and $p^0 = \Phi(c^0, t^0, \zeta^0)$ for some $c^0 \in \overline{Q_c}$, $\|t^0\| = \sigma$, $\zeta^0 \in \Omega' \cap U$.

For simplicity we may assume that $t^0 = (0, \dots, 0, \sigma)$ and set $'c = (c_1, \dots, c_{n-1})$, $'t = (t_1, \dots, t_{n-1})$. From (4.8) it follows also, that $c \in Q_c$.

We consider the transformation

$$(4.9) \quad S('c, 't, \zeta) = \Phi('c, c_n^0, 't, t_n^0, \zeta) : 'Q \times \overline{U} \longrightarrow \mathbb{C}^n,$$

where $'Q := \{z \in Q : c_n = c_n^0, t_n = t_n^0\} \subset \mathbb{R}^{2n-2}$.

Then $S('c^0, 't^0, \zeta^0) = p^0$ and its Jacobian is given by

$$J('c, 't, \zeta) = \hat{J}(c, t, \zeta)|_{c_n=c_n^0, t_n=t_n^0},$$

where

$$\hat{J}(c, t, \zeta) = \begin{vmatrix} \frac{\partial X_1}{\partial c_1} & \cdots & \frac{\partial X_{n-1}}{\partial c_1} & \frac{\partial Y_1}{\partial c_1} & \cdots & \frac{\partial Y_{n-1}}{\partial c_1} & \frac{\partial X_n}{\partial c_1} & \frac{\partial Y_n}{\partial c_1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{\partial X_1}{\partial c_{n-1}} & \cdots & \frac{\partial X_{n-1}}{\partial c_{n-1}} & \frac{\partial Y_1}{\partial c_{n-1}} & \cdots & \frac{\partial Y_{n-1}}{\partial c_{n-1}} & \frac{\partial X_n}{\partial c_{n-1}} & \frac{\partial Y_n}{\partial c_{n-1}} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{\partial X_1}{\partial t_1} & \cdots & \frac{\partial X_{n-1}}{\partial t_1} & \frac{\partial Y_1}{\partial t_1} & \cdots & \frac{\partial Y_{n-1}}{\partial t_1} & \frac{\partial X_n}{\partial t_1} & \frac{\partial Y_n}{\partial t_1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{\partial X_1}{\partial t_{n-1}} & \cdots & \frac{\partial X_{n-1}}{\partial t_{n-1}} & \frac{\partial Y_1}{\partial t_{n-1}} & \cdots & \frac{\partial Y_{n-1}}{\partial t_{n-1}} & \frac{\partial X_n}{\partial t_{n-1}} & \frac{\partial Y_n}{\partial t_{n-1}} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{\partial X_1}{\partial \zeta'} & \cdots & \frac{\partial X_{n-1}}{\partial \zeta'} & \frac{\partial Y_1}{\partial \zeta'} & \cdots & \frac{\partial Y_{n-1}}{\partial \zeta'} & \frac{\partial X_n}{\partial \zeta'} & \frac{\partial Y_n}{\partial \zeta'} \\ \frac{\partial X_1}{\partial \zeta''} & \cdots & \frac{\partial X_{n-1}}{\partial \zeta''} & \frac{\partial Y_1}{\partial \zeta''} & \cdots & \frac{\partial Y_{n-1}}{\partial \zeta''} & \frac{\partial X_n}{\partial \zeta''} & \frac{\partial Y_n}{\partial \zeta''} \end{vmatrix},$$

Here $\zeta = \zeta' + i\zeta''$ and $Y_k(c, t, \zeta) = h_k^* \circ X(c, t, \zeta) + t_k v(\zeta)$, $k = 1, \dots, n$.

The determinant J , is composed by 9 block matrices D_{ij} , $i, j = 1, 2, 3$.

We will show that $J({}'c^0, {}'t^0, \zeta^0) \neq 0$, which will imply that the operator S is a local diffeomorphism in a neighborhood of the point $({}'c^0, {}'t^0, \zeta^0)$.

Indeed, by (4.3) $X(c, 0, \zeta) \equiv c$, $h^*(c, 0, \zeta) \equiv h(c)$ and then

$$\begin{vmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{vmatrix}_{(c, 0, \zeta)} = D_{11} \cdot D_{22} = v^{n-1}(\zeta)$$

and

$$(4.10) \quad \begin{vmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{vmatrix}_{(c, t, \zeta)} = v^{n-1}(\zeta) + O(\varepsilon),$$

where we recall that $\varepsilon = \sup \{ \|c\| + \|t\| : c \in Q_c, t \in Q_t \}$. Note also that

$$D_{33} = \begin{vmatrix} \frac{\partial X_n}{\partial \zeta'} & \frac{\partial Y_n}{\partial \zeta'} \\ \frac{\partial X_n}{\partial \zeta''} & \frac{\partial Y_n}{\partial \zeta''} \end{vmatrix} = \left| \frac{d}{d\zeta}(X_n + iY_n) \right|^2.$$

Now consider the right hand side near the arc γ . It is clear that for every $s > 0$, there is an open set $\tilde{\Omega} \supset \gamma_0$ such that

$$(4.11) \quad \left| \frac{d}{d\zeta}(X_n + iY_n)(c, t, \zeta) \right|^2 \geq |D_\tau X_n(c, t, \tau)|^2 - s, \forall \zeta \in U \cap \tilde{\Omega}, \tau \in \gamma_0.$$

We calculate $D_\tau X(c, t, \tau)$, for $(c, t) \in Q$, $\tau \in T$,

$$(4.12) \quad D_\tau X(c, t, \tau) = -D_\tau \Im h \circ X(c, t, \tau) - t D_\tau \Im v(\tau)$$

Since, $D_\tau \Im v(\tau) = D_{\vec{n}} v(\tau)$, where $D_{\vec{n}}$ is the normal derivative \vec{n} , then (4.12) implies

$$D_\tau X(c, t, \tau) + t D_{\vec{n}} v(\tau) = \Im D_\tau h \circ X(c, t, \tau).$$

For k -coordinate of vector $X(\tau) = X(c, t, \tau)$ we have

$$(4.13) \quad \begin{aligned} & \left\| D_\tau X_k(c, t, \tau) + t_k D_{\vec{n}} v(\tau) \right\| = \|\Im D_\tau h_k \circ X(c, t, \tau)\| \leq \\ & \leq \text{const} \|D_\tau h_k \circ X(c, t, \tau)\| \leq O(\varepsilon) \|D_\tau X(c, t, \tau)\|. \end{aligned}$$

Therefore,

$$(4.14) \quad \begin{aligned} & \left| t_k D_{\vec{n}} v(\tau) \right| - O(\varepsilon) \|t\| \leq |D_\tau X_k(c, t, \tau)| \leq \\ & \leq \left| t_k D_{\vec{n}} v(\tau) \right| + O(\varepsilon) \|t\|, \quad 1 \leq k \leq n, \quad \tau \in T \end{aligned}$$

The second part of (4.14) implies

$$(4.15) \quad \|D_\tau X(c, t, \tau)\| \leq C \|t\|, \quad (c, t, \tau) \in Q \times T, \quad C - \text{constant}$$

As in Lemma 4.1 if $b = \inf_{\gamma_0} \left| D_{\vec{n}} v(\tau) \right| > 0$ and $O(\varepsilon) < \frac{b}{2}$, then the first part of (4.14) implies

$$(4.16) \quad |D_\tau X_k(c, t, \tau)| \geq |t_k| b - \|t\| b/2,$$

for $\tau \in \gamma, 1 \leq k \leq n$.

By (4.10) and (4.11) it follows that

$$\begin{aligned} |J('c, 't, \zeta)| &= |D_{11}| \cdot |D_{22}| \cdot \left| \frac{d}{d\zeta} (x_n + iy_n)('c, 't, \zeta) \right|^2 + O(\varepsilon) \\ &\geq [v^{n-1}(\zeta) + O(\varepsilon)] \cdot [|t_n b/2|^2 - s] + O(\varepsilon), \end{aligned}$$

for all $('c, 't, \zeta) \in 'Q \times [\mathbb{U} \cap \Omega']$, because $\|t^0\| = |t_n^0|$.

We can take $\tilde{\Omega} \cap \Omega'$ instead of Ω and observe that all functions $O(\cdot)$ do not depend on ζ . Therefore if we take ε, s small enough, then $|J('c^0, 't^0, \zeta^0)| > 0$.

Since, the plane $\{t_n = t_n^0\}$ is tangent to the sphere $\|t\| = \sigma$ at the point t^0 , the Jacobian of the restriction $\check{S} = \Phi('c, c_n^0, \sqrt{|t_1|^2 + \dots + |t_{n-1}|^2}, \zeta)$ also is not zero at the point $('c^0, 't^0, \zeta^0)$. In particular, the operator

$$\check{S} : U_1 \times U_2 \times U_3 \rightarrow U(p)$$

is a homeomorphism, where $U_1 \subset \mathbb{R}^{n-1}$ — a neighborhood of the point $'c^0$, $U_2 \subset Q'_t$ — a neighborhood of $t^0 \in Q'_t$ and $U_3 = \{|\zeta - \zeta_0| < \sigma'\} \subset \Omega, \sigma' > 0$, is a neighborhood of ζ_0 . It follows, that the open set $U(p) \subset W$, that is contradiction to $p \in \partial W$. \square

5. PROOF OF THE MAIN THEOREM

First we observe that from the results of Edigarian-Wiegerinck [EW10] and the authors [SZ12], it follows that $M \subset \mathbb{C}^n$ is a pluriregular set. Indeed, it was proved in [SZ12] that a set of full measure in a generic manifold M is not thin. Since the set $P = \{z \in M : V_M^*(z) > 0\}$, where $V_M^*(z)$ is Green function, is pluripolar by Bedford and Taylor ([BT82]), it has zero-measure (see [Sa76, C92]) and then the set $M \setminus P$ is not thin. Therefore $V_M^* \equiv 0$ on M , i.e. M is pluriregular. Note that in [EW10] non-thinness of $M \setminus P$ was

proved for C^1 -smooth manifold M and for a pluripolar set $P \subset M$, which implies that an arbitrary C^1 -smooth generic manifold is pluriregular.

Our main theorem will be a consequence of the following result, thanks to Lemma 2.3.

Theorem 5.1. *Any C^2 -smooth generic submanifold $M \subset \mathbb{C}^n$ is Λ_1 -pluriregular.*

Proof. We first reduce to the case of a totally real submanifold. Fix a point, say $z^0 = 0 \in M$. Changing holomorphic coordinates in \mathbb{C}^n , we can assume that the tangent space T_0M , which by definition does not contain any complex hyperplane, can be written as

$$T_0M = \{z = x + iy \in \mathbb{C}^n : y_1 = \cdots = y_{2n-m} = 0\}.$$

Hence for a small neighborhood $G = G_1 \times G_2$ of the origin with

$$G_1 = \{(x, y'') = (x, y_{2n-m+1}, \dots, y_n) \in \mathbb{R}^n \times \mathbb{R}^{m-n} : |x| \leq \delta, |y''| < \delta\},$$

$$G_2 = \{y' = (y_1, \dots, y_{2n-m}) \in \mathbb{R}^{2n-m} : |y'| < \delta\},$$

we can represent M as a graph

$$M \cap G = \{z \in G : y' = h(x, y'')\},$$

where h is C^2 smooth mapping from G_1 into G_2 .

Observe that for each small enough y''_0 the intersection $M \cap \Pi\{y''_0\}$ of M with the plane $\Pi\{y''_0\} := \{z \in \mathbb{C}^n : y'' = y''_0\}$ is an n -dimensional generic manifold. Moreover, since the Green function is monotonic, i.e. $V(z, E_1) \geq V(z, E_2)$ for $E_1 \subset E_2$, it is enough to prove the theorem in the case when M is generic of dimension n , hence totally real of dimension n .

In this case, we show the local Hölder pluriregularity of M , using previous results from Section 4. Fix a point $p \in M$, a ball $B(p) = B_x \times B_y \subset \mathbb{C}^n$ centered at the point p such that $M_p := M \cap B(p)$ is the graph of a C^2 -smooth function. Then by Corollary 4.2, for arbitrary fixed small $\sigma > 0$ there exist a neighborhood $\Omega' \supset \gamma_0$ and a constant $C > 0$, depending on the point p , such that the inequalities (4.7) hold. By Lemma 4.3 $O(p) \ni p$, where $O(p) = W^0$ is the interior of the set $W = W(p, \Omega', \sigma)$, constructed in Lemma 4.3.

Fix a point $z^0 \in O(p) \setminus M$ and a disk $\Phi(c, t, \zeta) : \Phi(c^0, t^0, \zeta^0) = z^0$, with $c^0 \in \overline{Q}_c$, $t^0 \in Q'_t$, $\zeta^0 \in \mathbb{U} \cap \Omega'$. Then the function $V_{M_p} \circ \Phi(c^0, t^0, \zeta) \in sh(\mathbb{U})$ and $V_{M_p} \circ \Phi|_\gamma \equiv 0$. Let $C'' = \max_{B(p)} V_{M_p}(z) < \infty$. By the theorem of two

constants we have

$$(5.1) \quad V_{M_p} \circ \Phi(c^0, t^0, \zeta) \leq C''[\omega^*(\zeta, \gamma, \mathbb{U}) + 1], \quad \zeta \in \mathbb{U}.$$

Therefore the first part of Lemma 3.1 and (4.7) yields the inequality

$$(5.2) \quad \begin{aligned} V_{M_p}(z^0) &= V_{M_p} \circ \Phi(c^0, t^0, \zeta^0) \leq C'' [\omega^*(\zeta^0, \gamma, \mathbb{U}) + 1] \leq \\ &\leq C' C'' d_{\mathbb{C}}(\zeta^0, \gamma_0) \leq C_p d_{\mathbb{C}^n}(z^0, M_p), \end{aligned}$$

for all $z^0 \in O(p)$, where $C_p := CC'C''$ depends on the fixed point $p \in M$ and on the corresponding family of analytic discs, attached to M locally, in a neighborhood of p .

Now given a compact set $K \subset M$ we can apply the previous estimate to each point of K . Then by compactness we can find a finite number of points p_1, \dots, p_k of K , a finite number of balls $B(p_1), \dots, B(p_k)$ and a finite numbers of open sets $O(p_1), \dots, O(p_k)$ such that

$$V_{M_p}(z) \leq C_p d_{\mathbb{C}^n}(z, M_p),$$

for any $z \in O(p)$ and $p = p_1, \dots, p_k$. Now observe that $O = \cup_{1 \leq i \leq k} O(p_i)$ is a neighborhood of K and shrinking a little bit the open sets O_p we can assume that for any $p = p_i$ and $z \in O_p$, $d_{\mathbb{C}^n}(z, M_p) \leq d(z, M)$. Since $V_M \leq V_{M_p}$, it follows that $V_M(z) \leq A d_{\mathbb{C}^n}(z, K)$, for any $z \in O$. \square

6. OPEN PROBLEMS

Let $D \subset M$ a domain with C^1 -smooth boundary ∂D . Lemma 4.2 states that a neighborhood of the generic manifold locally consists in the interior \dot{W} of the set $W = \{\Phi(c, t, \zeta) : c \in \overline{Q_c}, t \in Q'_t, \zeta \in \Omega = \overline{U} \cap \overline{\Omega'}\}$. It seems clear, at least intuitively, that if here, instead of Ω , we take its part $\Omega_{a,\alpha}$, $\alpha > 0$ (see Lemma 3.1), then we should see that \dot{W} contains some wedge

$$\{z \in \mathbb{C}^n : d_{\mathbb{C}^n}(z, M) < C_\alpha \cdot d_{\mathbb{C}^n}(z, \partial D)\},$$

where $C_\alpha > 0$ is a constant. If this is true then we could prove: *arbitrary close C^1 -domain \overline{D} in C^2 -smooth generic manifold is pluriregular, i.e. the Green function $V^*(z, \overline{D})$ is continuous in \mathbb{C}^n* . The proof easily follows by the well-known criteria of pluriregularity (see [Sa80]) and by the following lemma.

Lemma 6.1. *If $f(\lambda)$ is a C^1 -smooth function on $[0, 1] \subset \mathbb{R}$, then for every $\varepsilon > 0$ there exist polynom $p(\lambda)$ such that*

$$|p(\lambda) - f(\lambda)| \leq \varepsilon \lambda, \quad \lambda \in [0, 1].$$

The authors do not know any proof of the following

Conjecture: Let $D \subset M$ be a bounded domain in M with smooth boundary. then $\overline{D} \subset \mathbb{C}^n$ is $\Lambda_{1/2}$ -pluriregular i.e. its pluricomplex Green function $V(z, \overline{D})$ is Hölder continuous of order $1/2$ in \mathbb{C}^n .

We note that if M is real analytic generic manifold then the conjecture is true.

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